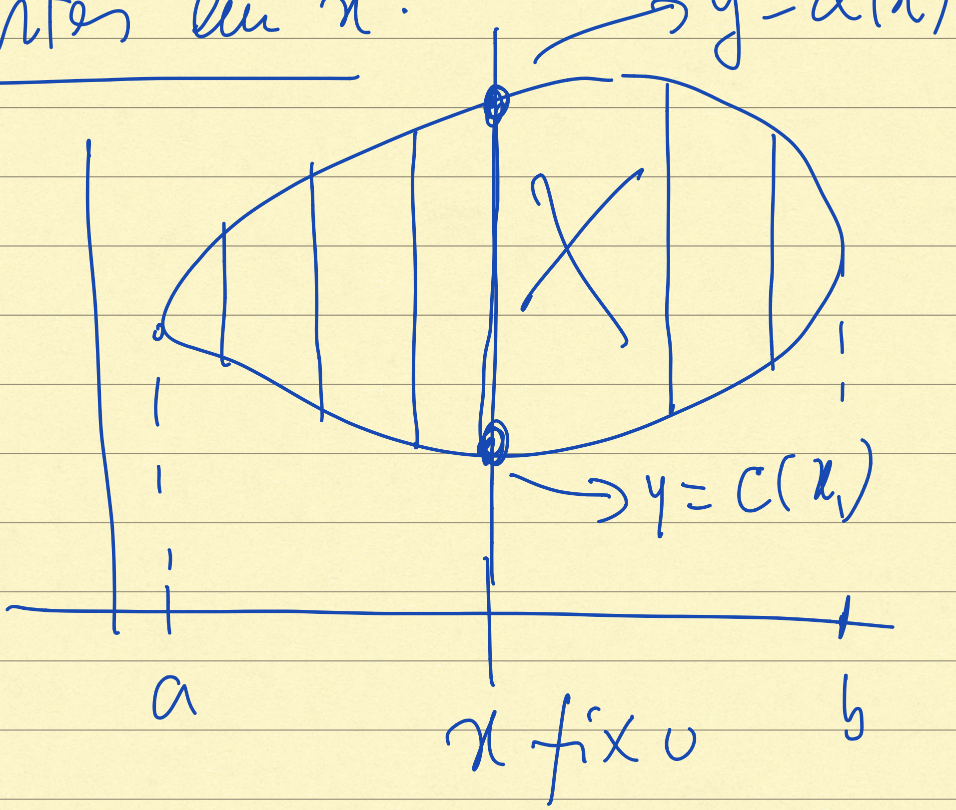
Integrais Múltiplos. Área. Volume

Área em $\mathbb{R}^2$: (Integrais duplos)

1) $dy\,dx$ $\qquad \displaystyle\int_a^b \left(\int_{c(x)}^{d(x)} 1\, dy \right) dx$

Cortes em x.



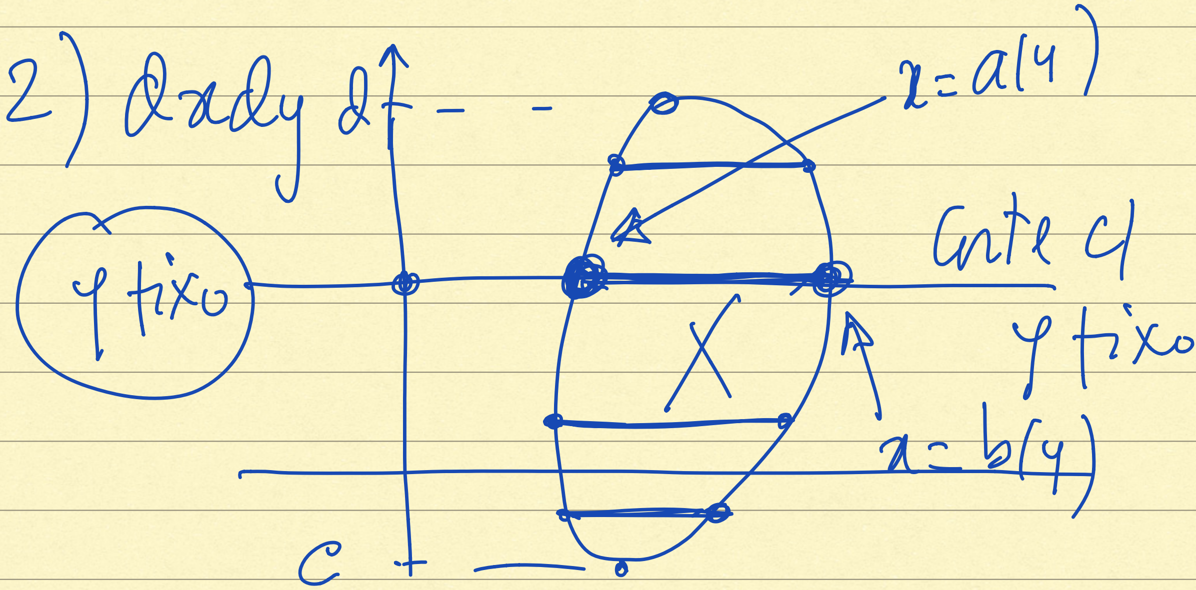
Área de $X \equiv vol_2(X)$

(Volume de dimensão 2 de $X \subset \mathbb{R}^2$)

$$Vol_2(X) = \int_a^b \left(\int_{c(x)}^{d(x)} dy \right) dx$$

$$= \int_a^b \underbrace{\left(d(x) - c(x) \right)}_{\text{Comprimento do corte em } x.} dx$$

2) $dx\,dy$

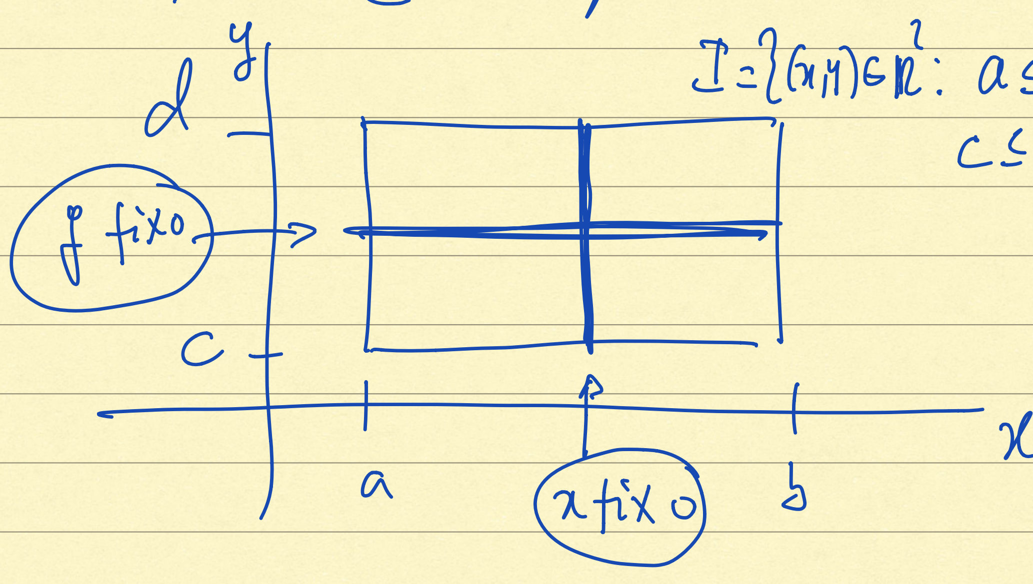


$$vol_2(X) = \int_c^d \left(\int_{a(y)}^{b(y)} dx \right) dy$$

$$= \int_c^d \underbrace{\left(b(y) - a(y) \right)}_{\text{comprimento do corte em } y.} dy$$

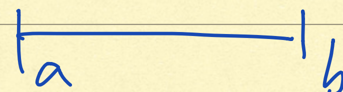
Exemplo: (simples) $\boxed{\text{intervalo}}$

$$I = \{ (x,y) \in \mathbb{R}^2 : a \leq x \leq b \, ; \, c \leq y \leq d \}$$



$$\mathrm{Vol}_2(X) = (b-a)\times(d-c)$$
$$= (d-c)\times(b-a)$$

$$(b-a)\times(d-c) = (b-a)\int_c^d 1\,dy$$
$$= \int_c^d (b-a)\,dy$$



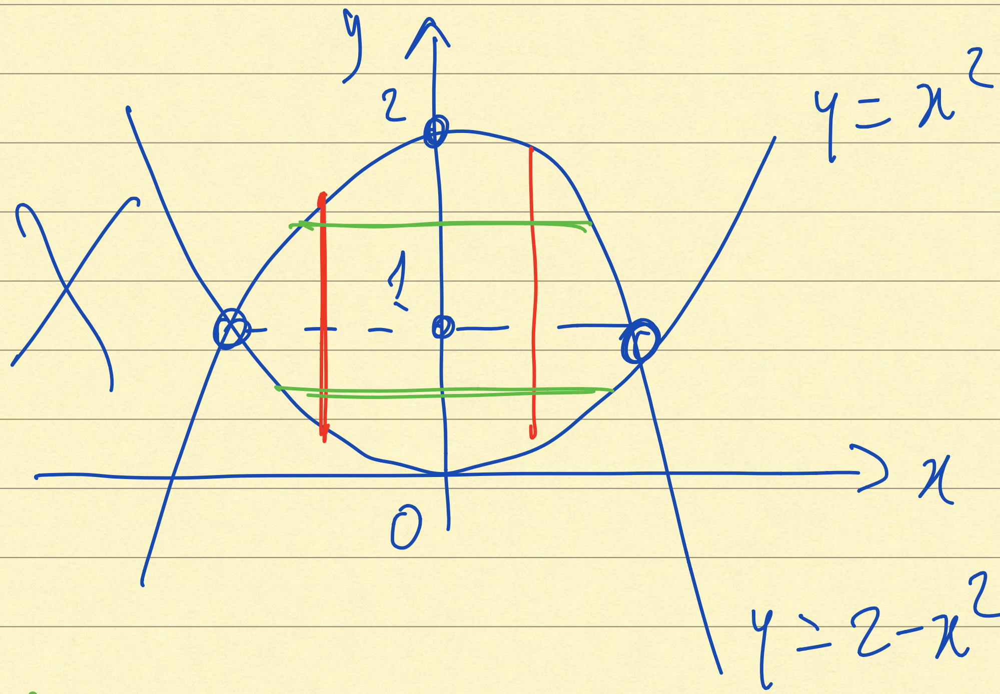
$$= \int_c^d\left(\int_a^b dx\right)dy = \mathrm{Vol}_2(X)$$

$$(d-c)\times(b-a) = (d-c)\int_a^b dx$$
$$= \int_a^b (d-c)\,dx = \int_a^b\left(\int_c^d dy\right)dx$$

$$\iint_X dx\,dy \;\equiv\; \iint_X dy\,dx$$

$\boxed{\text{FUBINI}}$

<u>Exemplo</u>: $X = \{(x,y) \in \mathbb{R}^2 : x^2 \le y \le 2 - x^2\}$



$dx\,dy \longrightarrow$ soma de 2 integrais duplos

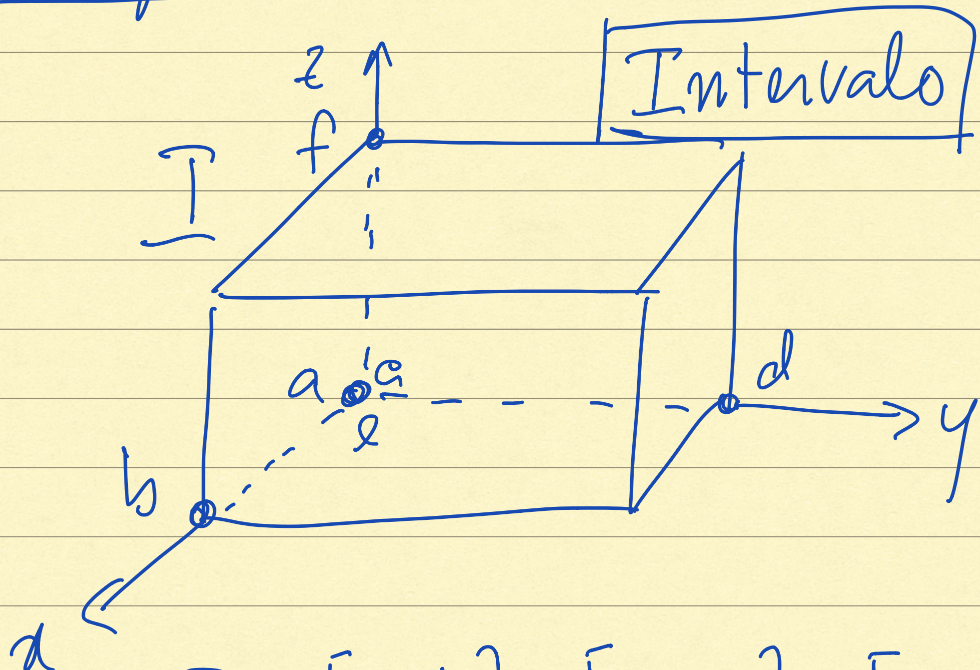
$dy\,dx \longrightarrow$ 1 integral duplo

Volume em $\mathbb{R}^3$: $\quad X \subset \mathbb{R}^3$

Volume de $X \equiv vol_3(X)$

$\equiv$ $\boxed{\text{integral de áreas}}$

Exemplo simples



$$\underline{I} = [a,b] \times [c,d] \times [e,f]$$

$$I = \{(x,y,z) \in \mathbb{R}^3 : a \leq x \leq b;\ c \leq y \leq d;\ e \leq z \leq f\}$$

$a, b, c, d, e, f \in \mathbb{R}.$

$$\mathrm{Vol}_3(I) = \underset{\substack{\uparrow \\ \text{comprimento}}}{\overline{(b-a)}} \times \underset{\substack{\uparrow \\ \text{largura}}}{(d-c)} \times \underset{\substack{\uparrow \\ \text{altura}}}{(f-e)}$$

6 maneiras $\neq$s $\Rightarrow$ <u>o mesmo resultado</u>

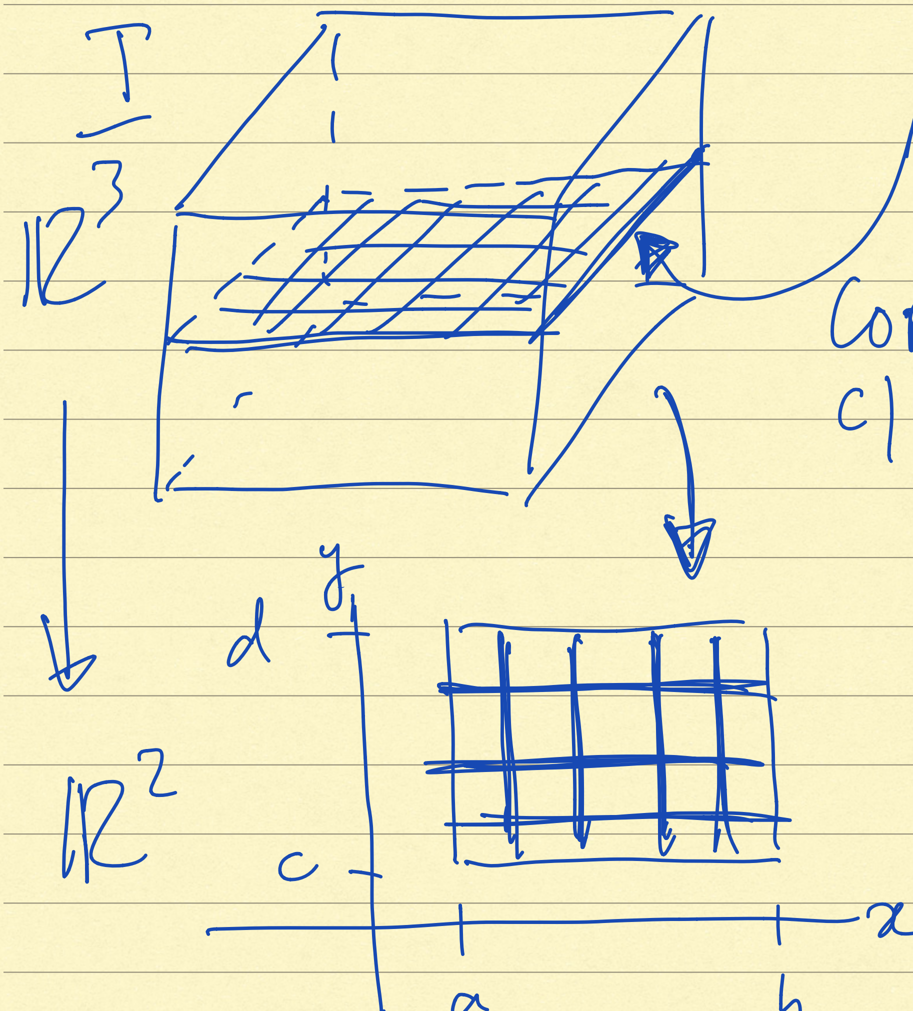
$$(b-a)\times(d-c)\times(f-e) =$$

$$= (b-a)\times(d-c)\int_e^f dz = (b-a)\int_e^f (d-c)\,dz$$

$$= (b-a)\int_e^f\left(\int_c^d dy\right)dz = \int_e^f\left(\int_c^d (b-a)\,dy\right)dz$$

$$= \int_e^f \left(\int_c^d \left(\int_a^b dx \right) dy \right) dz$$

$dx\,dy\,dz$ $\qquad \boxed{z \text{ fixo}}$

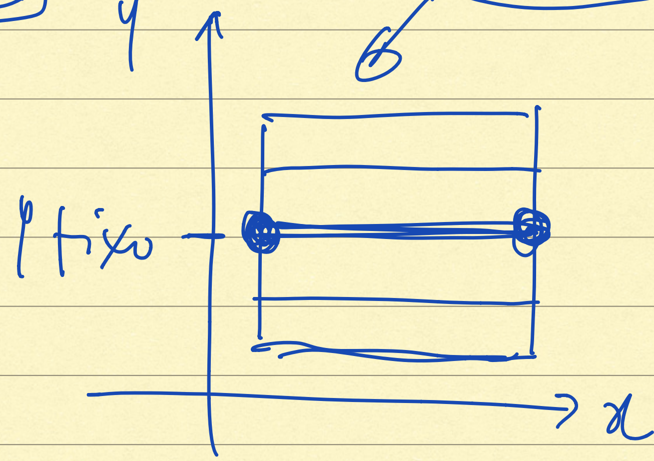


plano horizontal

CORTE em I c/ z fixo

"fatia" do bolo I.

$\boxed{dx\,dy}\,dz$ — z fixo



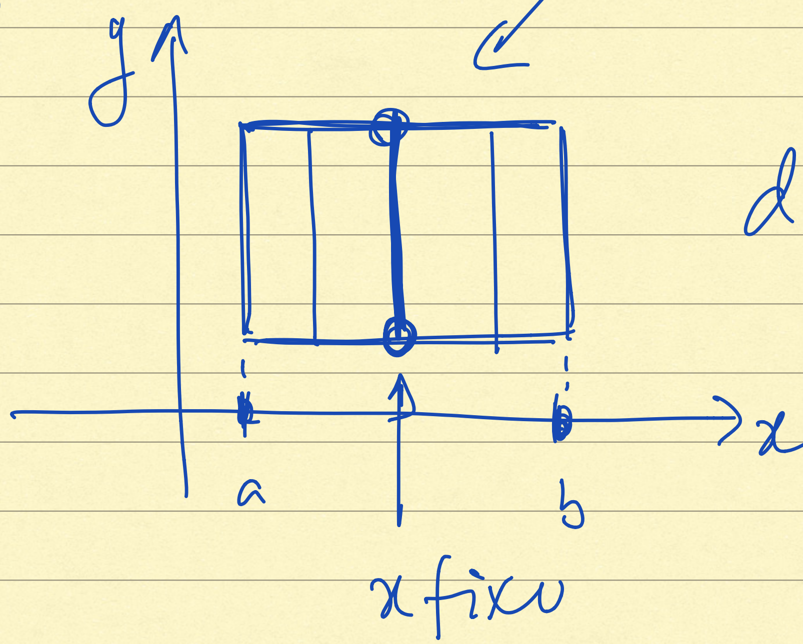
$dx\,dy$

y fixo

ou

$\underbrace{dy\,dx}\,dz$ — $\boxed{z \text{ fixo}}$

$\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$



$dy\,dx$

6 maneiras $\exists \int$:

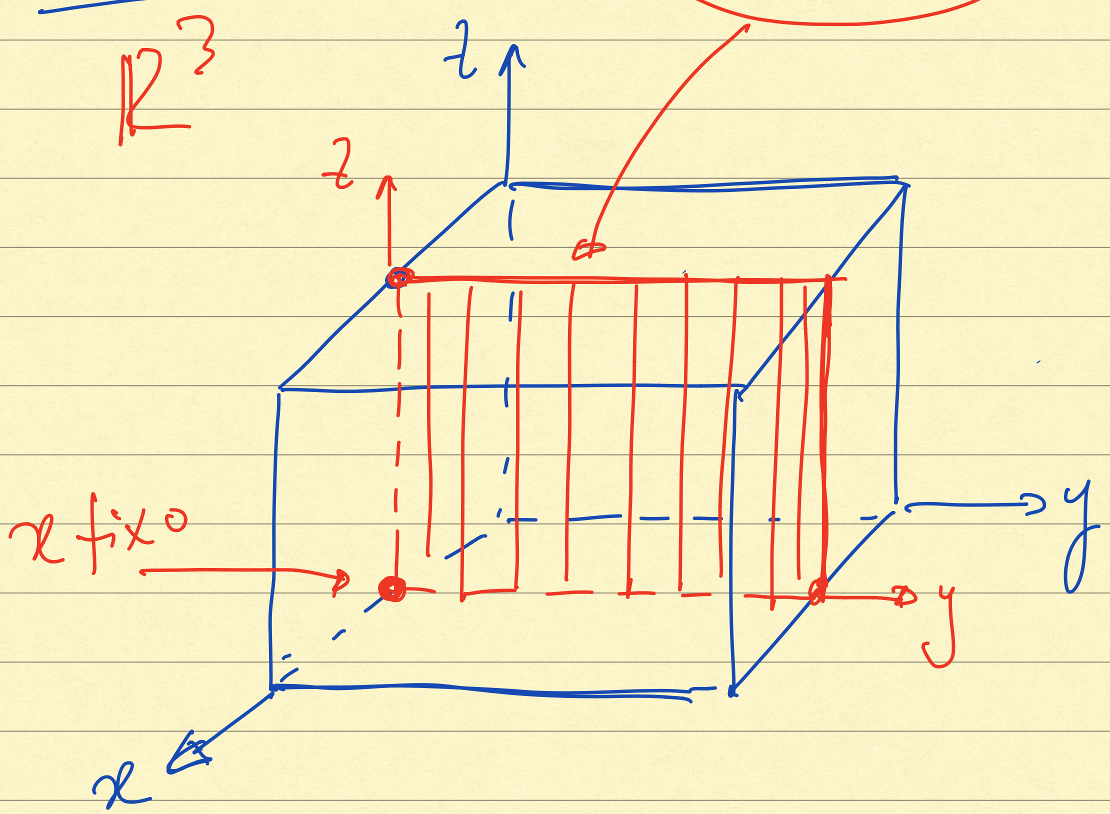
$dx\,dy\,dz$, $dy\,dx\,dz$

$dx\,dz\,dy$; $dz\,dx\,dy$

$dy\,dz\,dx$; $dz\,dy\,dx$

$\boxed{dz\,dy\,dx} \longrightarrow$ x fixo

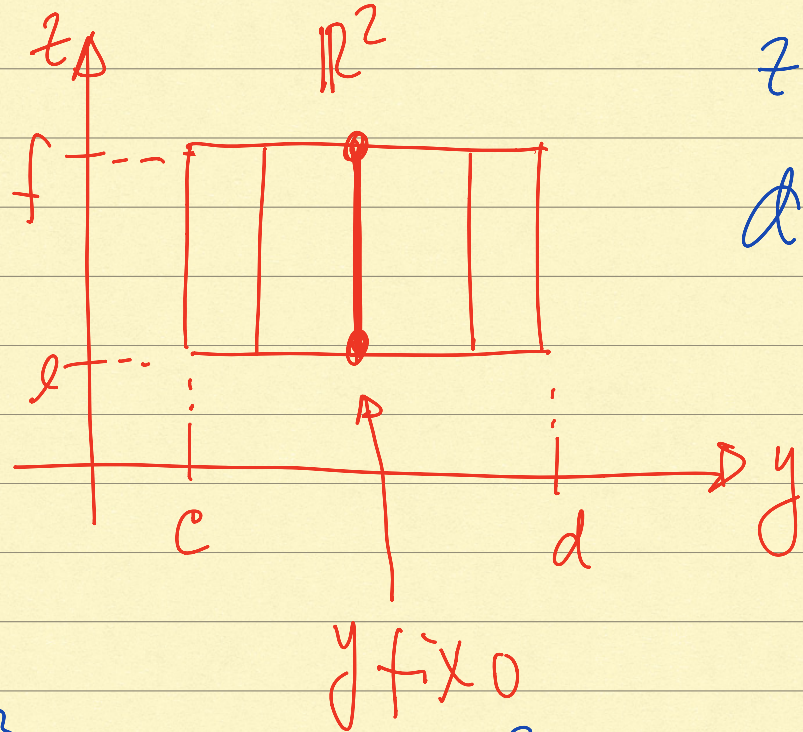
$\mathbb{R}^3$



"fatia"

CORTE

c/ x fixo



z fixo

$\underbrace{dz\,dy}_{y \text{ fixo}}$

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$dz\,dy\,dx \longrightarrow x$ fixo $\longrightarrow y$ fixo

$\longrightarrow$ calcular integral em z.

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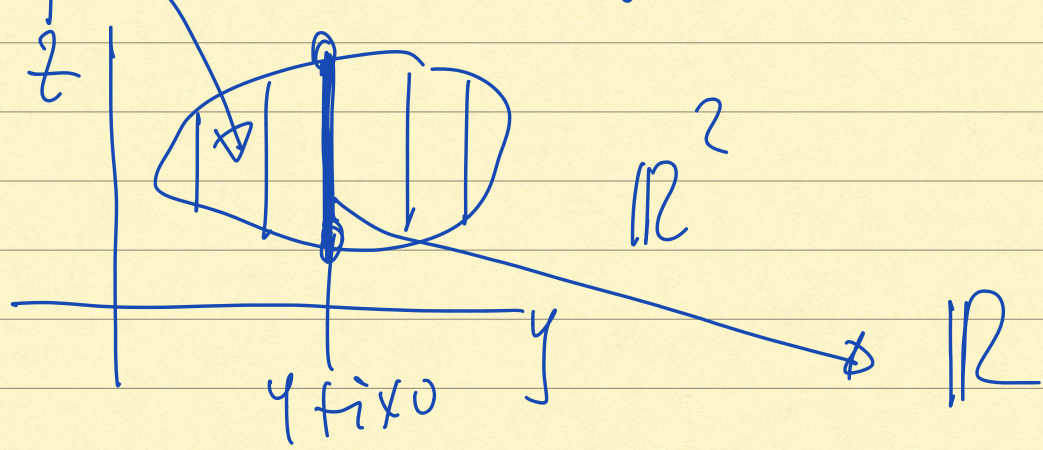
x fixo $\longrightarrow y$ fixo

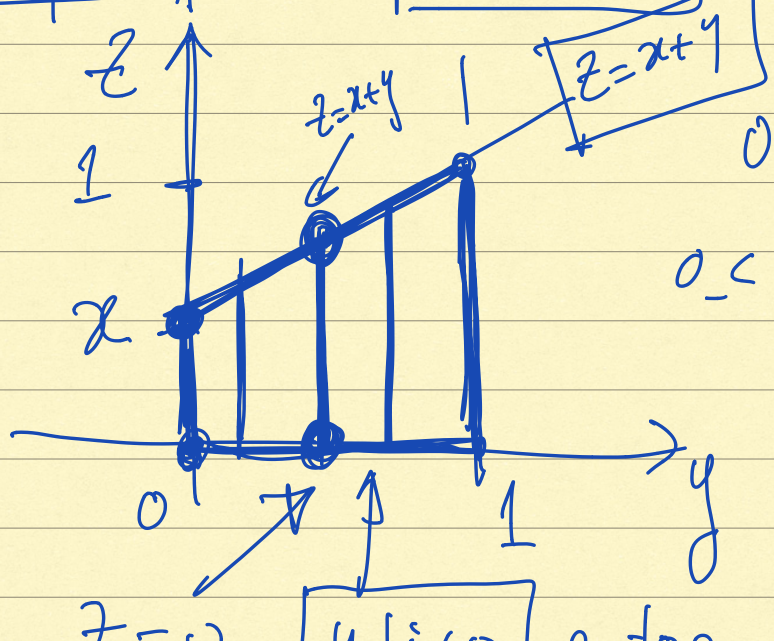


Exemplo:

$$X = \{ (x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1 ;\ 0 \le y \le 1 ;\ 0 \le z \le x+y \}$$

$\underbrace{dz\,dy\,dx}$ ou $dz\,dx\,dy$

$\boxed{x \text{ fixo}} \longrightarrow \boxed{0 \le x \le 1}$



$0 \le y \le 1$

$0 \le z \le x+y$

$\boxed{y \text{ fixo}}$ entre $0 \le y \le 1$

$$\mathrm{Vol}_3(X) = \int_0^1 \left(\int_0^1 \left(\int_0^{x+y} dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^1 (x+y)\, dy \right) dx$$

$$= \int_0^1 \left(x + \frac{1}{2} \right) dx$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$dz\,dx\,dy \longrightarrow$ fácil

$\boxed{dx\,dy\,dz}$

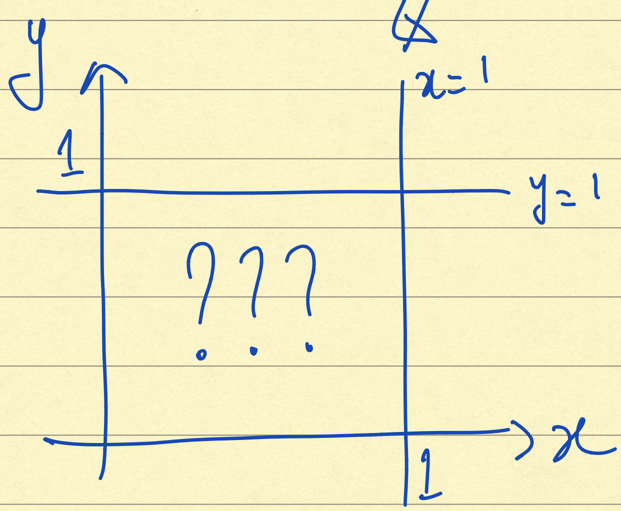
$\boxed{z \text{ fixo}}$

$0 \le x \le 1$

$0 \le y \le 1$

$0 \le z \le x+y$

$\boxed{0 \le z \le 2}$



$\boxed{x+y \geq z}$

?? $\boxed{x+y = z}$

por exemplo:

$x+y=1$